Logarithmic equivalence of Welschinger and Gromov-Witten invariants

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To the memory of Andrey Bolibruch, a lively man of creative mind and open soul

Abstract

The Welschinger numbers, a kind of a real analog of the Gromov-Witten numbers which count the complex rational curves through a given generic collection of points, bound from below the number of real rational curves for any real generic collection of points. By the logarithmic equivalence of sequences we mean the asymptotic equivalence of their logarithms. We prove such an equivalence for the Welschinger and Gromov-Witten numbers of any toric Del Pezzo surface with its tautological real structure, in particular, of the projective plane, under the hypothesis that all, or almost all, chosen points are real. We also study the positivity of Welschinger numbers and their monotonicity with respect to the number of imaginary points.

1 Introduction

The present note is devoted to an asymptotic enumeration of real rational curves interpolating fixed real collections of points in a real surface Σ, more precisely, to the following question: *given a real divisor D and a generic collection $\boldsymbol{w}$ of $c_1(\Sigma) \cdot D - 1$ real points in Σ, how many of the complex rational curves in the linear system $|D|$ passing through $\boldsymbol{w}$ are real ?* By rational curves we mean irreducible genus zero curves and their degenerations, so that they form in $|D|$ a projective subvariety $S(\Sigma, D)$; this subvariety is called the *Severi variety*. A curve on a real surface Σ is called real, if it is invariant under the involution $c : \Sigma \to \Sigma$ defining the real structure of Σ.

While, under mild conditions on Σ and D, the number of complex curves in question is the same for all generic $\boldsymbol{w}$ (it equals to the degree of $S(\Sigma, D)$), it is no more the case for real curves (except few very particular situations). For example, if Σ is the projective plane, such a non-invariance manifests starting from the degree $d = 3$: if $d = 3$ and all the $3d - 1 = 8$ points are real, the number of interpolating real rational curves takes the values $8, 10$, and 12 (twelve is the number of complex interpolating curves).

Till recently, in this interpolation problem, even the existence of at least one real solution for arbitrary choice of a generic collection of $3d - 1$ points in $\mathbb{R}P^2$ was known only for $d \leq 3$. The situation has radically changed after the discovery by J.-Y. Welschinger [10, 11] of a way to attribute weights of ± 1 to real solutions so that the number of solutions counted with weights is independent of the configuration of points; more precisely, a real collection $\boldsymbol{w}$ may even contain complex conjugated pairs of points, and then the result depends only on the number m of pairs of imaginary points in $\boldsymbol{w}$, and d. As an immediate consequence, the absolute value of Welschinger's invariant $W_{d,m}$ provides a lower bound on the number $R_{d,m}(\boldsymbol{w})$ of real solutions: $R_{d,m}(\boldsymbol{w}) \geq |W_{d,m}|$.

In [3] we proved an inequality $W_{d,0} \geq \frac{1}{2}d!$ which, due to Welschinger's lower bound, implies that *for any integer $d \geq 1$, through any $3d - 1$ generic points in $\mathbb{R}P^2$ there can be traced at least $\frac{1}{2}d!$ real rational curves of degree d.*

Comparing this lower bound with the number N_d of complex plane rational curves of degree d passing through $3d - 1$ generic points one can observe that the logarithm of the bound is asymptotically equal to $\frac{1}{3} \log N_d$. In fact, $\log d! \sim d \log d$, and $\log N_d \sim 3d \log d$, as follows from the inequalities $(3d - 4)! \cdot 54^{-d} \leq N_d \leq (3d - 5)!$ which in their turn follow from Kontsevich's recurrent formula [4] (a more precise asymptotics is found in [1], cf. Remark (3) in 2.4 below). The next natural question arises: *how far are $W_d = W_{d,0}$ from N_d in the logarithmic scale?*

In the present note we give a complete answer to the above question.

Theorem 1   *The sequences $\log W_d$ and $\log N_d$ are asymptotically equivalent.*

To establish this asymptotic equivalence we improve the lower bound $W_{d,0} \geq \frac{1}{2}d!$ we obtained in [3]. As in [3], the method we use is based on Mikhalkin's approach [6] to counting nodal curves passing through specific configurations of points, an approach which deals with a corresponding count of tropical curves. In fact, we slightly modify the construction of [3] in order to multiply the logarithm of the bound by 3.

We obtain similar results for curves on other toric Del Pezzo surfaces equipped with their tautological real structure. We also study the behavior of the double sequence $W_{d,m}$ with respect to m. To treat $W_{d,m}$ with $m > 0$ we use Shustin's counting scheme, see [8], which extends Mikhalkin's scheme from pure real data ($m = 0$) to arbitrary complex conjugation invariant ones. (It may be worth noticing, that these counting schemes can be considered as an advanced, almost explicit, version of the Viro patchworking, cf. [6, 7, 8, 9].)

We also note that our asymptotic results give some information on the convergency domain of the Gromov-Witten potential, see Remarks in 2.4.

The paper is organized as follows. A separate section, Section 2, is devoted to the case where all the points are real. The main results are summarized in Theorem 4 (which contains the above Theorem 1 as a particular case). The proof is divided in two parts: an upper bound for Gromov-Witten invariants (Lemma 5) and a lower bound for Welschinger invariants (Theorem 3). In Section 3 we analyze the

case where some of the points are imaginary. We start from explicit calculations of Welschinger invariants in few particular cases. Then, after resuming Shustin's general counting scheme, we derive from it few results on positivity, monotonicity, and asymptotics of Welschinger invariants, Theorems 7 and 8.

As is already mentioned, the counting scheme used in Section 2 is the same as in [3]. It is taken from [5, 7] and its summary is found in [3]. At the same time, it is included as a special case into Shustin's extended scheme described in Section 3, so that a reader can reconstruct the first scheme from the extended one specializing the number of imaginary points to zero.

2 Asymptotics of the Welschinger invariants for purely real data

<div style="border:1px solid black; width:100px; height:50px;"></div>

> He had bought a large map representing the sea,
> Without the least vestige of land:
> And the crew were much pleased when they found it to be
> A map they could all understand.
>
> THE HUNTING OF THE SNARK, LEWIS CARROLL

2.1 Notations

There are five *unnodal* (*i.e.*, not containing any (-2)-curve) toric Del Pezzo surfaces: the projective plane $\mathbb{P}^2$, the product of projective lines $Q = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^2$ with k blown up generic points, $k = 1, 2$ or 3; the latter three surfaces are denoted by P_k. Let $E_1, \ldots, E_k$ be the exceptional divisors of $P_k \to \mathbb{P}^2$ and $L \subset P_k$ the pull back of a generic straight line.

We equip $\mathbb{P}^2$ and Q with their tautological real and toric structures. For P_k, we choose the blown up points in $\mathbb{P}^2$ to be among the three 0-dimensional orbits, so that P_k inherits toric and real structures from $\mathbb{P}^2$.

Let D be an ample divisor on Σ. The linear system $|D|$ is generated, with respect to suitable real toric coordinates, by monomials $x^i y^j$, where (i, j) ranges over all the integer points (*i.e.*, points having integer coordinates) of a convex polygon $\Pi = \Pi_D$ of the following form. If $\Sigma = \mathbb{P}^2$ and $D = d[\mathbb{P}^1]$, then Π is the triangle with vertices $(0,0)$, $(d,0)$, and $(0,d)$. If $\Sigma = Q$ and D is of bi-degree (d_1, d_2), then Π is the

parallelogram with vertices $(d_2, 0)$, $(d_1 + d_2, 0)$, (d_1, d_2), and $(0, d_2)$. If $\Sigma = P_k$, $k = 1, 2, 3$, and $D = dL - \sum_{i=1}^{k} d_i E_i$, then Π is respectively the trapeze with vertices $(0, 0)$, $(d, 0)$, $(d_1, d - d_1)$, $(0, d - d_1)$, or the pentagon with vertices $(0, 0)$, $(d - d_1, 0)$, $(d - d_1, d_1)$, $(d_2, d - d_2)$, $(0, d - d_2)$, or the hexagon with vertices $(d_3, 0)$, $(d - d_1, 0)$, $(d - d_1, d_1)$, $(d_2, d - d_2)$, $(0, d - d_2)$, $(0, d_3)$. (A choice of a parallelogram instead of the rectangle in the case $\Sigma = Q$ simplifies the analysis of the irreducibility of the curves appearing in the proof of Theorem 3.)

Let $r = r(\Pi)$ be the number of integer points on the boundary $\partial\Pi$ of Π diminished by 1, and $\delta(\Pi)$ be the number of interior integer points of Π. Note that $r(\Pi) = c_1(\Sigma) \cdot D - 1$ and $\delta(\Pi)$ is the genus of nonsingular representatives of $|D|$. As is well known, the number of curves of genus $0 \le g \le \delta(\Pi)$ in $|D|$ passing through $c_1(\Sigma) \cdot D - 1 + g$ generic points is finite. Denote by $N_{\Sigma,D}$, or shortly N_D, the number of complex rational curves in $|D|$ passing through r given generic points of Σ (note that rational curves in $|D|$ which pass through r generic points are irreducible and nodal). Due to the genericity of the complex structure of unnodal Del Pezzo surfaces, these enumerative numbers coincide with the Gromov-Witten genus zero invariants. An inductive procedure reconstructing their values was given for $\Sigma = \mathbb{P}^2$ by M. Kontsevich and for other unnodal Del Pezzo surfaces by M. Kontsevich and Yu. Manin, see [4].

2.2 Welschinger numbers

Let us fix an integer m such that $0 \le 2m \le r$ and introduce a real structure $c_{r,m}$ on Σ^r which maps $(z_1, \ldots, z_r) \in \Sigma^r$ to $(z_1', \ldots, z_r') \in \Sigma^r$ with $z_i' = c(z_i)$ if $i > 2m$, and $(z_{2j-1}', z_{2j}') = (c(z_{2j}), c(z_{2j-1}))$ if $j \le m$. With respect to this real structure, a point $\boldsymbol{w} = (z_1, \ldots, z_r)$ is real, *i.e.*, $c_{r,m}$-invariant, if and only if z_i belongs to the real part $\mathbb{R}\Sigma$ of Σ for $i > 2m$ and z_{2j-1}, z_{2j} are conjugate for $j \le m$. In what follows we work with the open dense subset $\Omega_{r,m}(\Sigma)$ of $\mathbb{R}\Sigma^r = \text{Fix}\, c_{r,m}$ constituted of $c_{r,m}$-invariant r-tuples $\boldsymbol{w} = (z_1, \ldots, z_r)$ with pairwise distinct $z_i \in \Sigma$.

By abuse of language, we say that a curve C in Σ passes through $\boldsymbol{w} \in \Sigma^r$ if C contains all the components $z_i \in \Sigma$ of $\boldsymbol{w}$.

In the spirit of [10, 11], given a **generic** $\boldsymbol{w} \in \Omega_{r,m}(\Sigma)$, we introduce the number $W_{D,m}^{even}(\boldsymbol{w})$ (resp., $W_{D,m}^{odd}(\boldsymbol{w})$) of irreducible real rational curves in $|D|$ passing through $\boldsymbol{w}$ and having even (resp., odd) number of solitary nodes (*i.e.*, real double points, where a local equation of the curve can be written over $\mathbb{R}$ in the form $x^2 + y^2 = 0$). The *Welschinger number* $W_{D,m}(\boldsymbol{w})$ is defined by $W_{D,m}(\boldsymbol{w}) = W_{D,m}^{even}(\boldsymbol{w}) - W_{D,m}^{odd}(\boldsymbol{w})$.[1]

Theorem 2 (J.-Y. Welschinger, see [10, 11]). *The value $W_{D,m}(\boldsymbol{w})$ does not depend on the choice of a generic element $\boldsymbol{w}$ in $\Omega_{r,m}(\Sigma)$.*

In the case $m = 0$, the number $W_{D,m} = W_{D,m}(\boldsymbol{w})$ is denoted by W_D. We also use more detailed notations $W_{\Sigma,D}$ and $W_{\Sigma,D,m}$ when we need to work with several

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[1]In [3] we considered only the case $m = 0$ and used a slightly different notation.

surfaces Σ simultaneously.

2.3 Key bound

The following bound plays a crucial role in our treatment of logarithmic equivalence of Welschinger and Gromov-Witten invariants.

Theorem 3 *Let Σ be $\mathbb{P}^2$, Q, P_1, P_2, or P_3 equipped with its tautological real structure, and D an ample divisor on Σ. Then*

$$W_{nD} \geq \exp(a\,n\log n + O(n)), \quad n \in \mathbb{N}, \quad a = c_1(\Sigma) \cdot D . \tag{1}$$

2.4 Main corollary

Let as above Σ be $\mathbb{P}^2$, Q, P_1, P_2, or P_3 equipped with its tautological real structure, and D an ample divisor on Σ. We prove the following theorem.

Theorem 4 *The sequences $\log W_{nD}$ and $\log N_{nD}$, $n \in \mathbb{N}$, are asymptotically equivalent. More precisely,*

$$\log W_{nD} = \log N_{nD} + O(n) \quad and \quad \log N_{nD} = (c_1(\Sigma) \cdot D) \cdot n\log n + O(n). \tag{2}$$

Proof. Due to $N_{nD} \geq W_{nD}$, all the statements follow from Theorem 3 and Lemma 5. $\square$

Lemma 5 *The following inequality holds:*

$$\log N_{nD} \leq (c_1(\Sigma) \cdot D) \cdot n\log n + O(n). \tag{3}$$

Proof. The case of $\Sigma = Q$ with a divisor D of bi-degree (d_1, d_2) is equivalent to the case of $\Sigma = P_2$ with $D = dL - d_1 E_1 - d_2 E_2$, $d = d_1 + d_2$. The case of $\Sigma = \mathbb{P}^2$ is covered by [1]. Finally, one deduces the remaining case $\Sigma = P_k$, $D = dL - \sum_{i=1}^{k} d_i E_i$, from the case $\Sigma = \mathbb{P}^2, D = dL$, by means of the inequality

$$N_{P_k,D} \leq N_{\mathbb{P}^2,dL} \left(\prod_{i=1}^{k} d_k! \right)^{-1} \quad \text{if } c_1(P_k) \cdot D > 1 \tag{4}$$

applying (4) to nD, $n \geq 2$, instead of D, and using the Styrling formila for the factorial.

We prove (4) for P_k with any $k \in \mathbb{N}$ assuming, as usual, that the blown up points are in general position. Namely, we find a generic configuration $\boldsymbol{z}$ of

$$c_1(P_k) \cdot D - 1 = 3d - d_1 - ... - d_k - 1$$

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real points in P_k and a generic configuration $\mathbf{z}'$ of $3d - 1$ real points in $\mathbb{P}^2$ such that to any real rational curve in $|D|$ passing through $\mathbf{z}$, correspond $\prod_i d_i!$ real plane rational curves of degree d passing through $\mathbf{z}'$.

Let $\pi : P_k \to \mathbb{P}^2$ be the blowing-up at generic real points $p_1, ..., p_k \in \mathbb{P}^2$, and $\mathbf{z}$ a collection of $3d - d_1 - ... - d_k - 1$ generic real points in P_k. Since $c_1(P_k) \cdot D > 1$, Theorem 4.1(i) from [2] applies and it shows that any element in the set $\mathcal{R}_D(\mathbf{z})$ of rational curves in $|D|$, passing through $\mathbf{z}$, is an immersed curve. Furthermore, slightly moving one of the point of $\mathbf{z}$ all these elements become transversal to the exceptional divisors. Thus, they descend to plane rational curves of degree d having precisely d_i nonsingular branches at p_i for any $i = 1, \ldots, k$.

For each $i = 1, \ldots, k$, pick a collection $\mathbf{z}_i$ of d_i real generic points in a small neighborhood of p_i. For each $C \in \mathcal{R}_D(\mathbf{z})$, let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ be an immersion parametrizing $\pi(C)$. The normal bundle $\mathcal{N} = \phi^* T\mathbb{P}^2 / T\mathbb{P}^1$ is a line bundle of degree $3d - 2$ on $\mathbb{P}^1$, and the linear system $H^0(\mathcal{N})$ has no base points. Thus, imposing on deformations of ϕ the conditions to pass through all the $3d - d_1 - \cdots - d_k - 1$ elements of $\mathbf{z}$, one can freely vary the images of all the $d_1 + \cdots + d_k$ points of $\phi^{-1}\{p_1, \ldots, p_k\} \subset \mathbb{P}^1$, and therefore we obtain at least $\prod_i d_i!$ real rational curves passing through $\mathbf{z}' = \mathbf{z} \cup \bigcup_i \mathbf{z}_i$. $\qquad\square$

Remarks.

(1) Lemma 5 and its proof extend literally to symplectic blow-ups of $\mathbb{P}^2$ as soon as the numbers N_{nD} are replaced by the symplectic Gromov-Witten genus zero invariants.

(2) If $c_1(P_k) \cdot D = 1$, the inequality (4) can be replaced by

$$N_{P_k, D} \leq N_{\mathbb{P}^2, (d+1)L} \left(d \prod_{i=1}^{k} d_k! \right)^{-1}. \tag{5}$$

The set of rational curves in $|D|$ is finite, and to prove (5) one can choose as L any line transversal to all the rational curves C in $|D|$. Pick then three generic points z_1, z_2, z_3 on L, and a generic line $L' \neq L$ passing through z_3. A standard application of Riemann-Roch theorem shows that smoothing up any of the intersection points of L and C, one can include $C + L$ in a one-dimensional family of rational curves in $|D + L|$ passing through z_1, z_2, and that this family sweeps a neighborhood of z_3 in L'. Therefore, there exist at least $dN_{P_k, D}$ rational curves in $|D + L|$ passing through z_1, z_2 and a generic point $z_3' \in L'$ close to z_3. Now, it remains to apply (4) to $D + L$ instead of D.

(3) The results of this section provide some information on convergency properties of the Gromov-Witten potential, which in the case of $\Sigma = P_k$ we write as the Laurent series

$$f = \sum_{c_1(P_k) \cdot D \geq 1} \frac{N_D}{(c_1(P_k) \cdot D - 1)!} x^d y_1^{d_1} \cdots y_k^{d_k}, \quad D = dL - \sum d_i E_i.$$

First of all, the inequalities (4) and (5) imply that the convergency domain of f has a nonempty interior. More precisely, the convergency domain contains the set

$$\{|y_1| + \cdots + |y_k| < 1, \ |x|(1 + |y_1| + \cdots + |y_k|)^3 < a^{-1}\},$$

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where $a = 0,138\ldots$ is the radius of convergency of $\sum \frac{N_d}{(3d-1)!}t^d$ (see [1]). Indeed, f is term by term bounded from above, except the finite number of (Laurent) terms corresponding to $D = E_i$, $1 \le i \le k$, by

$$\sum_{c_1 \cdot D \ge 1, (d_1, \ldots, d_k) \ge 0} \frac{d^{-1}N_{d+1}}{\prod(d_k)! \cdot (3d - (d_1 + \cdots + d_k) - 1)!} x^d y_1^{d_1} \cdots y_k^{d_k}$$

$$= \sum_{d > 0} \frac{d^{-1}N_{d+1}}{(3d-1)!}(1 + y_1 + \cdots + y_k)^{3d-1}x^d.$$

Furthermore, from the lower bound

$$\log N_{nD} \ge \log W_{nD} \ge (c_1(\Sigma) \cdot D) \cdot n \log n + O(n)$$

it follows that the convergence domain of f is contained in the intersection of sets

$$\{|x^d y_1^{d_1} \cdots y_k^{d_k}| \le R(d, d_1, \ldots, d_k)\},$$

where $R(d, d_1, \ldots, d_k)$ are some positive finite constants, $(d_1, \ldots, d_k) \ge 0$ and $c_1(P_k) \cdot D \ge 1$. This assertion is a routine corollary of the following Cauchy formula for subdiagonals of a convergent power series $f = \sum a_{k\hat{l}} x^k y_1^{l_1} \cdots y_k^{l_k}$ in $k + 1$ variables $x, \hat{y}$ ($\hat{y} = (y_1, \ldots, y_k)$):

$$\sum_n a_{pn,\hat{q}n} t^{pn} = \frac{1}{(2\pi i)^k} \int_T f(\frac{t}{\hat{y}^{\hat{q}}}, \hat{y}^p)\frac{d\hat{y}}{\hat{y}}, p \in \mathbb{N}, \hat{q} \in \mathbb{N}^k,$$

where $|t|^p \le x^p(0)\hat{y}(0)^{\hat{q}}$, T is a torus $\{|y_i|^p = y_i(0)\}$, and $p, \hat{q}$ are coprime. This integral formula holds for any positive $(x(0), \hat{y}(0))$ in the interior of the convergency domain. The boundedness of $x^p\hat{y}^{\hat{q}}$ is thus equivalent to finiteness of the convergency radius of the subdiagonal, and the latter follows from the above lower bound.

2.5   Proof of the Key Bound (Theorem 3)

We use the same terminology, notations, and constructions as in [3]. The only difference concerns the subdivisions called compressing in [3]; here, we prefer to follow the terminology from general Shustin's scheme 3.3.1 and call them *consistent subdivisions*, or *γ-consistent subdivisions* when we want to underline that the subdivisions are obtained by the compressing procedure applied to a fixed path γ.

We consider, first, the cases of $\mathbb{P}^2$ and Q, and then show how to deduce from them the required statements in all the remaining cases.

2.5.1   Case $\Sigma = \mathbb{P}^2$

It is sufficient to treat $D = [\mathbb{P}^1]$.

Let us fix $n \in \mathbb{N}$ and consider a linear function $\lambda^0 : \mathbb{R}^2 \to \mathbb{R}$ defined by $\lambda^0(i, j) = i - \varepsilon j$, where $\varepsilon > 0$ is sufficiently small constant (so that λ^0 defines a kind of a lexicographical order on the integer points of the triangle $\Pi = \Pi_{nD}$). Inscribe in
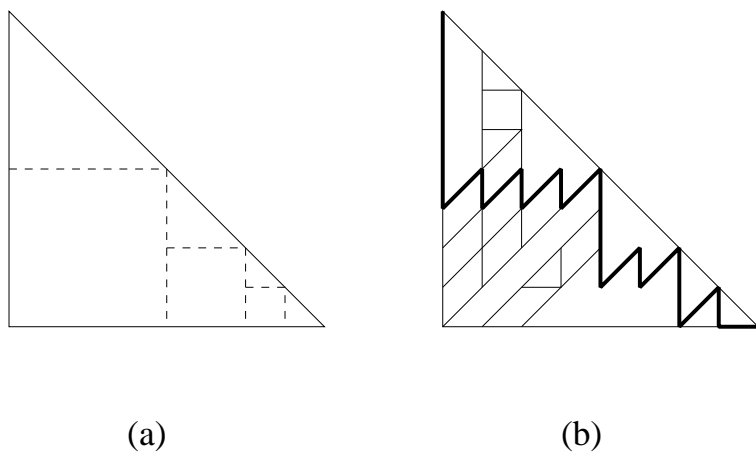
(a)   (b)

Figure 1: Path γ and γ-consistent subdivisions for $\Sigma = \mathbb{P}^2$

Π a sequence of maximal size squares as shown on Figure 1(a). Their right upper vertices have the coordinates

$$(x_i, y_i), \ i \geq 1, \ x_1 = y_1 = \left[\frac{n}{2}\right], \ y_{i+1} = \left[\frac{n - x_i}{2}\right], \ x_{i+1} = x_i + y_{i+1}.$$

Put $(x_0, y_0) = (0, n)$. Then pick a λ^0-admissible lattice path γ consisting of segments of integer length 1 as shown on Figure 1(b). This path consists of sequences of vertical segments joining (x_i, y_i) with $(x_i, y_{i+1} - 1)$, zig-zag sequences joining $(x_i, y_{i+1} - 1)$ with (x_{i+1}, y_{i+1}) (in such a zig-zag sequence the segments of slope 1 alternate with vertical segments; it always starts and ends with segments of slope 1), and the segments $[(n - 1, 1), (n - 1, 0)]$ and $[(n - 1, 0), (n, 0)]$. The total integer length of this lattice path is $3n - 1$.

Now, we select some γ-consistent subdivisions of Π, that is subdivisions constructed along the compressing procedure (see Subsection C in [3]) starting with γ. Namely, the path γ divides Π in two parts: the part $\Pi_+(\gamma)$ bounded by γ and $\partial\Pi_+$ (in our case, $\partial\Pi_+$ is the hypotenuse of the triangle Π), and the part $\Pi_-(\gamma)$ bounded by γ and $\partial\Pi_-$ (two other sides of the triangle). Subdivide $\Pi_+(\gamma)$ in vertical strips of integer width 1. Note that the rightmost strip consists of one primitive triangle. Pack into each strip but the rightmost one the maximal possible number of primitive parallelograms and place in the remaining part of the strip two primitive triangles (see Figure 1(b)). Then subdivide $\Pi_-(\gamma)$ in slanted strips of slope 1 and horizontal width 1. Pack into each strip the maximal possible number of primitive parallelograms. This gives a subdivision of any slanted strip situated above the line $y = x$. For any strip situated below the line $y = x$ place in the remaining part of the strip one primitive triangle (see Figure 1(b)).

The total number of such γ-consistent subdivisions is

$$M_n \geq \prod_i \frac{y_i!(y_i + 1)!}{2^{y_i}} \cdot \prod_i y_i! , \tag{6}$$

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where the first product corresponds to subdivisions of $\Pi_+(\gamma)$ and the second one to those of $\Pi_-(\gamma)$.

All the constructed subdivisions of Π are nodal and odd (see [3] for definitions), each of them is dual to an irreducible tropical curve and contributes 1 to the Welschinger number. For example, the irreducibility of the dual tropical curve can easily proved by the following induction. Let us scan the subdivision by vertical lines from right to left. The rightmost fragment of the tropical curve is dual to the primitive triangle $\Pi \cap \{x \geq n - 1\}$, so it is irreducible. At the i-th step, $i > 0$, we look at the irreducible components of the curve dual to the union of those elements of our subdivision which intersect the strip $n - i - 1 < x < n - i$. Each of these irreducible components either connects the lines $x = n - i - 1$ to $x = n - i$, or contains a pattern dual to a trianlge with an edge on $x = n - i$, or contains a pattern dual to a slanted parallelogram. Therefore, each component joins the curve dual to the subdivision of $\Pi \cap \{x \geq n - i\}$.

According to Proposition 2.5 from [3], any chosen subdivision contributes 1 to the Welschinger invariant, and according to Proposition 2.6 from [3] for our choice of the function λ^0, the input of any other consistent subdivision is nonnegative. Therefore, $W_n \geq M_n$ and it remains to check that

$$\log M_n \geq 3n \log n + O(n).$$

To that purpose, we observe that according to the relations defining the coordinates x_i, y_i the following inequalities hold: $y_i \leq n - x_i \leq 2y_{i+1} + 1$. They imply

$$y_{i+1} \geq 2^{-i}(y_1 + 1) - 1 \geq \frac{n}{2^{i+1}} - 1,$$

where i varies from 0 to $k \geq [\log_2 n] - 1$. Then we use the Styrling relation

$$\log \Gamma(x) = x \log x + O(x), \ x \to +\infty,$$

to get finally

$$\log \prod y_i! \geq \sum_{i=1}^{[\log_2(n)]} \log \Gamma(\frac{n}{2^i} - 1) = \sum_{i=1}^{[\log_2(n)]} \frac{n}{2^i} \log \frac{n}{2^i} + O(n)$$

$$= \sum_{i=1}^{\infty} \frac{n}{2^i} \log n + O(n) = n \log n + O(n).$$

$\square$

2.5.2 Case $\Sigma = Q$

Assume that $d_1 \geq d_2$. Fix n and consider the linear function $\lambda^0 : \mathbb{R}^2 \to \mathbb{R}$ introduced in 2.5.1. Put $a = [nd_2/2]$. In the parallelogram $\Pi = \Pi_{nD}$ pick a λ^0-admissible lattice
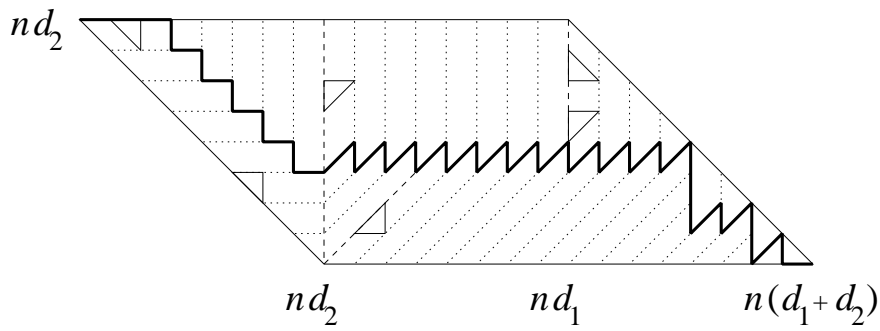
Figure 2: Compressing subdivisions for $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$

path γ consisting of segments of integer length 1 as shown on Figure 2. This path consists of the sequence of horizontal segments joining $(0, nd_2)$ with $(a-1, nd_2)$, a zig-zag sequence joining $(a-1, nd_2)$ with $(nd_2, a-1)$ (in this zig-zag sequence the vertical segments alternate with horizontal ones; it starts with a vertical segment and ends with a horizontal one), a zig-zag sequence joining $(nd_2, a-1)$ with $(nd_1 + nd_2 - a, a)$ (in this zig-zag sequence the segments of slope 1 alternate with vertical segments; it starts and ends with segments of slope 1), and the sequence of segments joining $(nd_1 + nd_2 - a, a)$ with $(nd_1 + nd_2, 0)$ and obtained by a translation of the path described in section 2.5.1 (one should replace n by a in this description). The total integer length of this lattice path is $2n(d_1 + d_2) - 1 = c_1(Q) \cdot nD - 1$.

Now, we select some compressing subdivisions of Π constructed along the compressing procedure starting with γ. Namely, subdivide $\Pi_+(\gamma)$ in vertical strips of width 1. Pack into each strip the maximal possible number of primitive parallelograms and place in the remaining part of the strip one or two primitive triangles. The number of these triangles is two if the strip is contained in the half-plane $x \geq nd_1$, and one if the strip lies between the vertical lines $x = nd_2$ and $x = nd_1$; the strip does not contain any triangle if the strip is contained in the half-plane $x \leq nd_2$.

The lines $x = nd_2$ and $y = x - nd_2$ divide $\Pi_-(\gamma)$ into three parts (see Figure 2). Subdivide these parts as follows:

- the left part is subdivided in horizontal strips of height 1; each strip is filled in by parallelograms and one primitive triangle (see Figure 2);

- the central part is subdivided in slanted strips of slope 1 and horizontal width 1; each strip is filled in by primitive parallelograms;

- the right part is subdivided in similar slanted strips; each strip is filled in by primitive parallelograms and one primitive triangle (see Figure 2).

Since each constructed subdivision contributes 1 in the Welschinger invariant and the other consistent subdivisions have nonnegative contributions, the required bound is obtained by computations completely similar to those performed in 2.5.1. $\square$
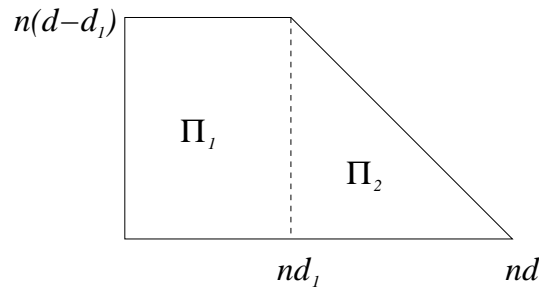
Figure 3: Case $\Sigma = P_1$

2.5.3 Other cases

We start with a reduction of the case $\Sigma = P_1$ to the case $\Sigma = Q$. Divide the trapeze $\Pi = \Pi_{nD}$ by the vertical line $x = n(d - d_1)$ into the rectangle Π_1 and the triangle Π_2. In the trapeze Π consider λ^0-admissible lattice paths γ of the form $\gamma = \gamma_1 \cup \gamma_2$, where

- γ_1 is any lattice path in Π_1 which starts at the left upper corner of Π_1, ends at the right lower corner of Π_1, consists of $2nd - 1$ segments of integer length 1, and can be completed to a compressing subdivision of Π_1 with a positive input to the Welschinger invariant;

- γ_2 is the lattice path which consists of $n(d - d_1)$ segments of length 1 and goes along the lower horizontal side of Π_2.

Since γ_1 ends at the right lower corner of Π_1, any γ_1-consistent subdivision of Π_1 can be completed to a γ-consistent subdivision of Π in the following way. Divide Π_2 into vertical strips of width 1. Pack in each strip the maximal possible number of primitive parallelograms, and put in the remaining part of each strip one primitive triangle. This can be done in $(n(d - d_1))!$ ways. Note that if a chosen γ_1-consistent subdivision of Π_1 is dual to an irreducible tropical curve, then the resulting γ-consistent subdivision of Π is dual to an irreducible tropical curve as well. Thus, W_{nD} is bounded from below by $W_{Q,(nd_1, n(d-d_1))} \cdot (n(d - d_1))!$. It implies the required result in the case $\Sigma = P_1$.

The case $\Sigma = P_2$ can be reduced to the case $\Sigma = Q$ in a similar way. The only difference is that the polygon Π is divided into a rectangle and a trapeze. The case $\Sigma = P_3$ is in its turn reduced to the case of $\Sigma = P_2$. □

3 Few results for data containing imaginary points

In this section we study the invariants $W_{d,m}$, $m > 0$, of toric Del Pezzo surfaces using as a basis the results of [8].

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3.1 Examples of calculation of Welschinger invariant

(1) All the plane rational cubics interpolating fixed $3d - 1 = 8$ points belong to the same linear pencil. The integration with respect to the Euler characteristic along the pencil gives the following **linear** function

$$W_{3,m} = 8 - 2m, \quad 0 \leq m \leq 4 .$$

Note, that whatever is $0 \leq m \leq 4$, the number $R_{3,m}(\boldsymbol{w})$ of real rational cubics attains the value $W_{3,m}$ for a suitable generic collection $\boldsymbol{w} \in \Omega_{r,m}, r = 3d - 1 = 8$. For example, the pencil spanned by the curves $x(x^2 + y^2 + z^2) + \epsilon y^3$ and $y(x^2 + y^2 + z^2) - \epsilon x^3$, where ϵ is sufficiently small, has 8 imaginary fixed points and does not contain real singular cubics (to check this property, it is sufficient to notice that if t is real, the polynomial $y^3 - tx^3$ cannot vanish at any intersection point of $x + ty = 0$ and $x^2 + y^2 + z^2 = 0$).

(2) To calculate the Welschenger invariant for plane quartics and quintics, we use birational transformations and Welschinger's wall crossing formula (see [11], Theorem 2.2) which expresses the first finite difference of the function $m \mapsto W_{D,m}$ as twice the Welschinger invariant of the surface Σ blown up at one real point. For plane quartics ($\Sigma = \mathbb{P}^2$, $D = 4L$) we obtain a system of difference equations

$$\Delta^2 W_{4,m} = 4W_{3,m} = 4(8 - 2m),$$
$$-\Delta^1 W_{4,m}|_{m=0} = 2W_{P^1 \times P^1, D=(2,3), m=0} = 2 \cdot 48 = 96,$$
$$W_{4,m}|_{m=0} = 240$$

(two last values are obtained by the algorithm described in [6, 7], see also [3]). For plane quintics we get similarly

$$-\Delta^3 W_{5,m} = 8W_{4,m},$$
$$\Delta^2 W_{5,m}|_{m=0} = 4W_{P^1 \times P^1, D=(3,3), m=0} = 4 \cdot 1086,$$
$$-\Delta^1 W_{5,m}|_{m=0} = 2W_{P^2(1), D=(5,2), m=0} = 9168,$$
$$W_{5,m}|_{m=0} = 18264.$$

Therefore, the Welschenger invariants of plane quartics take the values

m	0	1	2	3	4	5
W	240	144	80	40	16	0

which are interpolated by a polynomial of degree three

$$W_{4,m} = -\frac{4}{3}m(m-1)(m-2) + 16m(m-1) - 96m + 240.$$

For plane quintics the invariants take the values

m	0	1	2	3	4	5	6	7
W	18264	9096	4272	1872	744	248	64	64

which are interpolated by a polynomial of degree six

$$W_{5,m} = \frac{4}{45}m(m-1)(m-2)(m-3)(m-4)(m-5)-$$

$$-\frac{32}{15}m(m-1)(m-2)(m-3)(m-4) + 32m(m-1)(m-2)(m-3)-$$

$$-320m(m-1)(m-2) + 2172m(m-1) - 9168m + 18264.$$

In these two cases, $d = 4$ and 5, the degree of the interpolating polynomials happens to be smaller than a generic interpolation data, that is, smaller than $[\frac{3d-1}{2}]$. Let us note that it is no more the case for any $d \geq 6$.

(3) The Welschinger invariants for curves of bi-degree $(d, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ also can be computed explicitly. In this case, the number m of pairs of imaginary points varies in the range $0 \leq m \leq d+1$, and Welschinger's wall crossing formula takes the form

$$\Delta^1 W_{(d+1,2),m} = -2W_{(d,2),m}, \quad 0 \leq m \leq d . \tag{7}$$

A solution of this difference equation is uniquely determined, for instance, by the initial data $W_{(1,2),m} = 1, 1 \leq m \leq 3$, and the sequence $W_{(d,2),d+1}$, $d \geq 1$. We claim that

$$W_{(d,2),d+1} = \left[\frac{d+1}{2}\right] \cdot 2^{d-1}, \quad d \geq 1 , \tag{8}$$

and prove this formula in subsection 3.3.3. Thence, we obtain by induction all the other values:

$$W_{(d,2),m} = (d+m) \cdot 2^{2d-2-m}, \quad 0 \leq m \leq d .$$

For comparison, $W_{(2,d),0} = d \cdot 2^{2d-2}$ and, according to [1], $N_{(2,d)} = d(d+1) \cdot 2^{2d-3}$.

3.2 Two general results

The calculations of the Welschinger invariant presented in the previous subsection lead to the following conjecture.

Conjecture 6 *Let Σ be a toric Del Pezzo surface equipped with its tautological real structure, and D an ample divisor on Σ. The Welschinger invariants $W_{D,m}$ are positive if $m < [(c_1(\Sigma) \cdot D - 1)/2]$, are non-negative if $m = [(c_1(\Sigma) \cdot D - 1)/2]$, and satisfy the monotonicity relation*

$$W_{D,m-1} \geq W_{D,m} \quad for \quad m \geq 1 .$$

This conjecture is supported by the following two statements.

Theorem 7 *Let Σ be $\mathbb{P}^2$, P_1, P_2, or P_3 equipped with its tautological real structure, D an ample divisor on Σ, and m is a positive integer less or equal to $(c_1(\Sigma) \cdot D - 1)/2$. The invariants $W_{\Sigma,D,m}$ are positive for $m \leq 3$ in the case $\Sigma = \mathbb{P}^2$ and for $m \leq 2$ in the cases $\Sigma = P_k$, $k = 1, 2, 3$. Furthermore,*

$$\log W_{\Sigma,nD,m} = \log N_{\Sigma,nD} + O(n)$$

for $m \leq 3$ in the case $\Sigma = \mathbb{P}^2$ and for $m \leq 2$ in the cases $\Sigma = P_k$, $k = 1, 2, 3$. In addition, if Σ is $\mathbb{P}^2$, P_1, or P_2 and the projections of Π_D on the coordinate axes have at least the length three, then

$$W_{\Sigma,D,0} > W_{\Sigma,D,1} > W_{\Sigma,D,2}.$$

Theorem 8 *Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be equipped with its tautological real structure, and D be a divisor on Q of bi-degree (d_1, d_2) with positive d_1, d_2. Then, for any integer $1 \leq m < d_1 + d_2$, the Welschinger invariant $W_{Q,D,m}$ is positive. Furthermore, let a sequence of integers $m(n)$, $n \geq 1$, satisfy*

$$m(n) = \mu n + \psi(n), \quad 0 \leq m(n) < (d_1 + d_2)n,$$

where μ is some real number, and $\psi(n)$ is a sequence of real numbers such that $\lim_{n \to \infty} \frac{\psi(n)}{n} = 0$. Then

$$\log W_{Q,nD,m(n)} \geq (2d_1 + 2d_2 - \mu)n \log n + O(n + \psi(n) \log n) .$$

3.3   Proof of Theorems 7 and 8

In 3.3.1 and 3.3.2 we denote by $\Pi = \Pi_D$ the convex lattice polygon introduced in Section 2.1 and fix a non-negative integer m such that $2m \leq r = c_1(\Sigma) \cdot D - 1$.

3.3.1   Counting scheme

Here we shortly describe the way to compute $W_{D,m}$ worked out in [8] (notation in [8] is slightly different; there, $m = r''$ (the number of pairs of imaginary points in the point data) and $r - 2m = r'$ (the number of real points)).

Pick a linear function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ which is injective on the set $\Pi \cap \mathbb{Z}^2$, and choose a set $\mathcal{I} \subset \{1, 2, \ldots, r - m\}$ consisting of m elements. The counting goes through the following construction of subdivisions of Π into lattice triangles and parallelograms.

Let p and q be the vertices of Π such that

$$\lambda(p) = \min \lambda(\Pi), \quad \lambda(q) = \max \lambda(\Pi) .$$

A lattice path $\gamma \subset \Pi$ is called $(\lambda, \mathcal{I})$-*admissible*, if it has $r - m + 1$ vertices $v_0, \ldots, v_{r-m}$ such that

$$v_0 = p, \quad v_{r-m} = q, \quad \lambda(v_i) < \lambda(v_{i+1}), \quad i = 0, \ldots, r - m - 1 ,$$

and its $r - m$ *edges* $\sigma_i = [v_{i-1}, v_i]$, $i = 1, \ldots, r - m$, satisfy the following restrictions:

- if $i \notin \mathcal{I}$, then the integer length $|\sigma_i|$ of σ_i is odd,

- if $i \in \mathcal{I}$, then either $|\sigma_i|$ is even, or there exists an integer point $v \in \Pi \backslash \sigma_i$ satisfying the inequalities $\lambda(v_{i-1}) < \lambda(v) < \lambda(v_i)$.

Pick a $(\lambda, \mathcal{I})$-admissible lattice path $\gamma \subset \Pi$. The following recursive procedure describes the γ-*consistent subdivisions* of Π.

First, for any edge σ_i, $i \in \mathcal{I}$, of odd integer length, choose an integer point $v_i' \in \Pi \backslash \sigma_i$ with $\lambda(v_{i-1}) < \lambda(v_i') < \lambda(v_i)$ such that the integer length of $[v_{i-1}, v_i']$ and $[v_i', v_i]$ is odd. Then, put

$$S_0 = \gamma \cup \bigcup_i T_i,$$

where T_i is the triangle with the vertices v_{i-1}, v_i', v_i, and denote by τ_0 the subdivision of S_0 into segments σ_j and triangles T_i. We also put

$$W_{\lambda, \mathcal{I}}(\gamma, \tau_0) = (-1)^a \prod_i |T_i|,$$

where a is the number of integer points in the interior of S_0, and $|T_i|$ stands for the *integer* (*i.e.*, double Euclidean) *area* of T_i.

After that, we construct a sequence of contractible sets $S_k \subset \Pi$, $k \geq 0$, equipped with subdivisions τ_k into lattice segments, triangles, and parallelograms, and with numbers $W_{\lambda, \mathcal{I}}(\gamma, \tau_k)$, applying by induction the following rules:

- if S_k is convex, we stop the construction,

- if S_k is not convex, then among the vertices v of τ_k such that v is a common vertex of edges σ' and σ'' bounding an angle $< \pi$ outside S_k, we choose the vertex with the minimal value of λ,

- then we choose one of the two options: either take the parallelogram $P \subset \Pi$ generated by σ' and σ'', and put

$$S_{k+1} = S_k \cup P, \quad W_{\lambda, \mathcal{I}}(\gamma, \tau_{k+1}) = W_{\lambda, \mathcal{I}}(\gamma, \tau_k);$$

or take the triangle T with the sides σ' and σ'', unless $|\sigma'|$ and $|\sigma''|$ are odd and the length of the third side of T is even, and put

$$S_{k+1} = S_k \cup T, \quad W_{\lambda, \mathcal{I}}(\gamma, \tau_{k+1}) = W_{\lambda, \mathcal{I}}(\gamma, \tau_k) \cdot (-1)^{a(T)} \cdot A(T), \qquad (9)$$

where

$$a(T) = \begin{cases} \#(\mathrm{Int}(T) \cap \mathbb{Z}^2) + 1, & \text{if } |\sigma'| \text{ and } |\sigma''| \text{ are even}, \\ \#(\mathrm{Int}(T) \cap \mathbb{Z}^2), & \text{otherwise}, \end{cases}$$

$$A(T) = \begin{cases} 1, & \text{if } |T| \text{ is odd}, \\ |T| \cdot |\sigma'|^{-1}, & \text{if } |\sigma'| \text{ is even and } |\sigma''| \text{ is odd}, \\ 2|T| \cdot (|\sigma'| \cdot |\sigma''|)^{-1}, & \text{if } |\sigma'| \text{ and } |\sigma''| \text{ are even}, \end{cases}$$

- if none of the above options is performable, we stop the construction.

If this (compressing) procedure ends up with $S_k = \Pi$, it provides a subdivision τ_k which is convex, *i.e.*, consists of the linearity domains of some convex piecewise-linear function $\nu : \Pi \to \mathbb{R}$. This subdivision is dual to a simple tropical curve (see [6, 8]). If the curve is irreducible and if, in addition, the edges of τ_k which lie on $\partial\Pi$ have integer length ≤ 2, then the subdivision τ_k is called a γ-*consistent subdivision* of Π. Notice that the sets $S_0, ..., S_k$ can uniquely be restored out of $\mathcal{I}$, γ, and τ_k.

The main result of [8] states that

$$W_{D,m} = \sum_{\gamma,\tau} W_{\lambda,\mathcal{I}}(\gamma, \tau) \ , \tag{10}$$

where γ runs over all $(\lambda, \mathcal{I})$-admissible lattice paths in Π, and τ runs over all γ-consistent subdivisions of Π.

The number $W_{\lambda,\mathcal{I}}(\gamma, \tau)$ is called the *Welschinger coefficient* of τ.

In what follows, the edges of even integer length and the triangles with all the edges of even integer length are called *even*. The edges of odd integer length are called *odd*.

3.3.2   Even edges expansion and a sign formula

As in Section 2.5, let $\lambda^0 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\lambda^0(i, j) = i - \varepsilon j$, where $\varepsilon > 0$ is a sufficiently small constant. Fix a $(\lambda^0, \mathcal{I})$-admissible lattice path $\gamma \subset \Pi$.

Note that each non-vertical side of Π is parallel to a vector $(1, s)$ with $s = 0, \pm 1$. All lemmas of this subsection are, in fact, valid for more general polygons, namely for polygons whose non-vertical sides have integer *slopes*, that is, are parallel to vectors $(1, s)$ with $s \in \mathbb{Z}$.

Lemma 9 *If $\sigma = [(i, j), (i', j')]$, where $i' \geq i$, is an edge of a γ-consistent subdivision of Π, then*

$$i' - i \leq \begin{cases} 1, & \text{if } |\sigma| \text{ is odd,} \\ 2, & \text{if } |\sigma| \text{ is even.} \end{cases} \tag{11}$$

In particular, in any γ-consistent subdivision

- *each triangle has a vertical side;*

- *triangles with at least one odd side do not contain interior integer points;*

- *any triangle with a non-vertical even side is even.*

Proof. The three statements on triangles follow from the statement concerning the edges. The latter follows in its turn from the fact that the integer lengths of the boundary edges of a γ-consistent subdivision are at most 2. Indeed, if $\sigma \not\subset \partial\Pi$, consider the maximal k such that $\sigma \subset \partial S_k$ (see the construction in 3.3.1). Then S_{k+1} is obtained by adding a triangle or a parallelogram built on σ and an adjacent edge. Thus, ∂S_{k+1} contains an edge with the horizontal projection as long as that of σ and which, due to the compressing rules, has the same parity as σ. $\qquad\square$

We say that non-vertical edges σ^1, ..., σ^l of a γ-consistent subdivision τ of Π form a *chain* if, for any $1 \leq j \leq l-1$, the edges σ^j and σ^{j+1} are either parallel edges of a parallelogram of τ or belong to a triangle of τ which is not contained in S_0 (see section 3.3.1). If σ^l is contained in the boundary of Π, then the chain σ^1, ..., σ^l is called an *escaping chain* of σ^1.

Lemma 10 *Let τ be a γ-consistent subdivision of Π. Then*

- *any non-vertical edge σ_i of γ with $i \notin \mathcal{I}$ has exactly two escaping chains;*

- *any non-vertical even edge in γ has exactly two escaping chains;*

- *any non-vertical side of any triangle in S_0 has exactly one escaping chain;*

- *the parity of edges does not change along any chain;*

- *if σ^1, ..., σ^l is an escaping chain of σ^1 such that any edge σ^j with $j > 1$ does not belong to γ, then the slopes of edges do not increase along the chain;*

- *escaping chains of distinct edges of S_0 are disjoint, and any non-vertical edge of τ belongs to an escaping chain of some edge of S_0.*

Proof. All the statements can easily be derived from the compressing rules and Lemma 9. $\qquad\square$

Lemma 11 *Let τ be a γ-consistent subdivision of Π. Then the following sign formula holds*
$$\text{sign } W_{\lambda^0, \mathcal{I}}(\gamma, \tau) = (-1)^{\sum s(\sigma)},$$
where σ runs over all non-vertical even edges of γ, and $s(\sigma)$ stands for the difference of the slopes of the terminal edges of two escaping chains of σ.

Proof. According to Lemma 9, the triangles of odd integer area do not contain interior integer points. As to the number of interior integer points of an even triangle, it is equal to the half of the integer length of its vertical side diminished by one, since the length of the horizontal projection of a non-vertical even side is 2. Now, the required claim follows from (9). $\qquad\square$

Denote by $s_{\min}$ the minimum of slopes of non-vertical sides of Π.

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Lemma 12 *Let τ be a γ-consistent subdivision of Π. Then the slope of any non-vertical edge of τ is at least $s_{\min}$. Moreover, the slope of any non-vertical edge σ_i of γ with $i \notin \mathcal{I}$ and of any non-vertical even edge of γ is at least $s_{\min} + 1$.*

Proof. The both statements are immediate corollaries of Lemma 10. Indeed, a non-vertical edge of slope $< s_{\min}$ in τ cannot have an escaping chain. If an edge mentioned in the second statement has the slope $s_{\min}$, then the both escaping chains of this edge are formed by parallel edges. Thus, in this case, the corresponding tropical curve is reducible. $\qquad\square$

3.3.3 Example

As an example of application of the counting scheme given in 3.3.1, we evaluate the Welschinger invariant $W_{(2,d),d+1} = W_{D,d+1}$ for divisors D of bi-degree $(2,d)$, $d \geq 1$, on $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$.

Represent $|D|$ by the rectangle Π with vertices $(0,0), (d,0), (d,2), (2,2)$ and choose $\mathcal{I} = \{2, ..., d+2\} \subset \{1, ..., d+2\}$. Consider a $(\lambda^0, \mathcal{I})$-admissible lattice path γ generating a consistent subdivision of Π, and denote the consecutive $d+2$ edges of γ by σ_i, $1 \leq i \leq d+2$. Since $1 \notin \mathcal{I}$, the edge σ_1 is odd. Then, as it follows from Lemma 12, the edge σ_1 must be vertical, so of integer length 1. Lemma 12 implies as well that each of the other edges σ_i, $2 \leq i \leq d+2$,

- either has integer length 1, is of slope 0 or 1, and can be extended to a triangle of integer area 1 (as in the construction of S_0, section 3.3.1),

- or has integer length 2, and is vertical or of slope 1.

Therefore, γ starts with a sequence of segments of length 1 as shown in Figure 4(e); this sequence ends up on the upper side of Π. On the other hand, the total integer length of γ is $d+2$, and its displacement vector is $(d, -2)$; it implies that γ contains exactly one edge of slope 1 and integer length 1, and that the number of vertical even edges is greater by 1 than the number of slanted even edges. In addition, an even edge cannot be followed or preceded by a horizontal edge of integer length 1, since otherwise the subdivision would contain either an odd edge with horizontal projection of length 2, or an edge of negative slope (see Figures 4(a,b,c,d)). Hence the only remaining possibilities for γ are those shown in Figure 4(e), where $0 \leq k \leq (d-1)/2$.

Such a path γ admits 2^{d-2k-1} extensions up to S_0: to any horizontal edge one can attach a triangle either below or above (see Figure 4(e)). Further extension of S_0 to a γ-consistent subdivision of Π is unique. As a result, $\Pi \cap \{x \leq d - 2k\}$ becomes covered by parallelograms, triangles of integer area 1, and one triangle of integer area 2, while $\Pi \cap \{x \geq d - 2k\}$ is covered by triangles of integer area 4. The
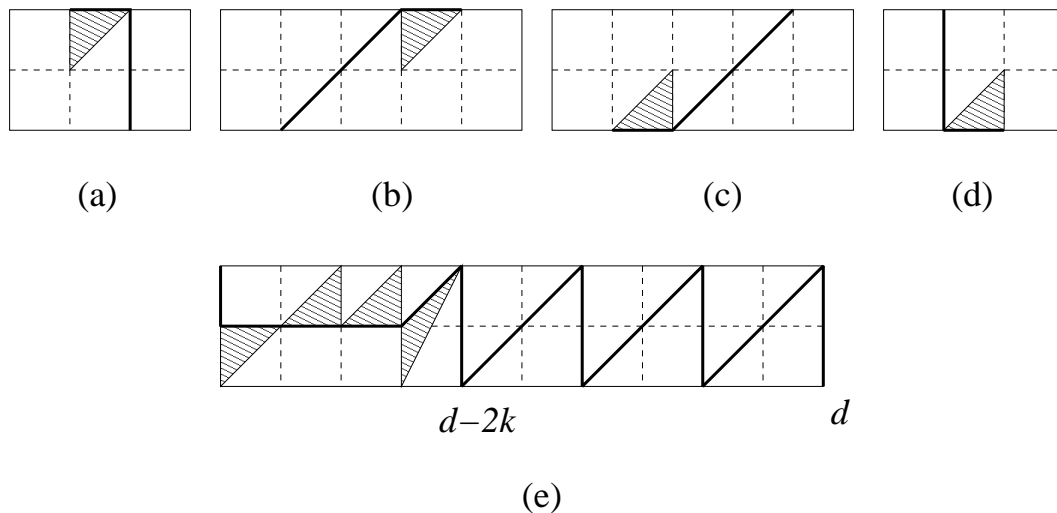
Figure 4: Computation of $W_{(2,d),d+1}$

Welschinger coefficient of the subdivision obtained is 2^{2k}. Hence,

$$W_{(2,d),d+1} = \sum_{0 \le k \le (d-1)/2} 2^{d-2k-1} \cdot 2^{2k} = \left[\frac{d+1}{2}\right] \cdot 2^{d-1} \ .$$

3.3.4   Proof of Theorem 7

First, we prove the positivity statement. More precisely, we show that, for some specific choices of $\mathcal{I}$, the Welschinger coefficients of all consistent subdivisions of Π are positive.

Assume that a $(\lambda^0, \mathcal{I})$-admissible lattice path γ contains at most one even edge. Then any γ-consistent subdivision τ does not contain any even triangle. Since, in addition, according to Lemma 9 the triangles with at least one odd edge in such a subdivision τ do not contain interior integer points, the positivity of the Welschinger coefficients of all γ-consistent subdivisions follows from the formula (9). This covers the case $m = 1$.

Let $m = 2$. Put $\mathcal{I} = \{1, r - m\}$, and assume that a $(\lambda^0, \mathcal{I})$-admissible lattice path γ has exactly two even edges (the other cases are covered by the previous consideration). Then the first and the last edges of γ are even. Denote these edges by l_1 and l_2. Let τ be a γ-consistent subdivision τ of Π with a negative Welschinger coefficient. Then τ has exactly one even triangle T. Two sides of T are parallel translations of l_1 and l_2, and the third side is non-vertical. It implies that either l_1 or l_2 is vertical, and thus is contained in the boundary of Π. In addition, the edges of the escaping chains of the non-vertical edge l_i belong to T and few parallelograms (see, for example, Figure 5(a)). Hence, the tropical curve dual to τ splits off a component consisting of three rays with a common vertex. This contradicts to the
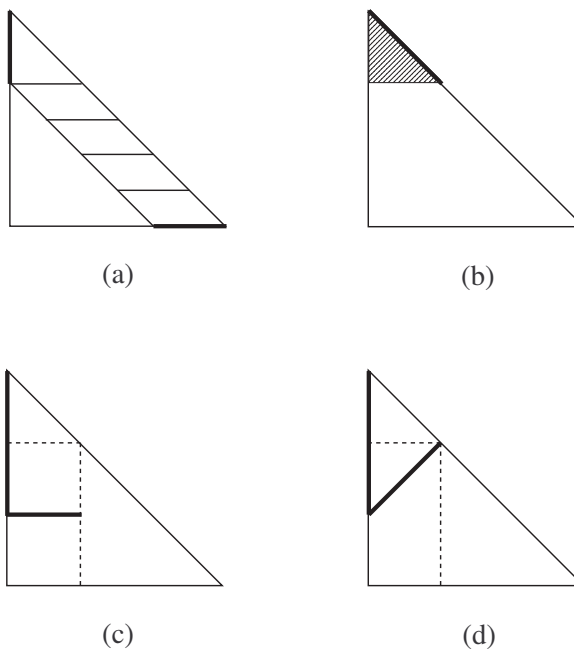
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Figure 5: Fragments of admissible paths with small m

consistency of τ (recall that one of the consistency conditions is the irreducibility of the dual tropical curve).

Let $m = 3$ and $\Sigma = \mathbb{P}^2$. Put $\mathcal{I} = \{1, 2, 3\}$. Consider a $(\lambda^0, \mathcal{I})$-admissible lattice path γ, and assume that some γ-consistent subdivision τ has a negative Welschinger coefficient. Then, by Lemma 11, the path γ has either one or three non-vertical even edges. In the latter case, τ does not have any even triangle, and according to Lemma 9 and the formula (9) the Welschinger coefficient of τ should be nonnegative. Hence, γ contains exactly one non-vertical even edge. Assume that the first edge of γ is odd. As it follows from Lemmas 9 and 12, this edge must be of integer length 1 and of slope -1; see Figure 5(b). The corresponding triangle T in S_0 has a vertical and a horizontal side. Lemma 10 implies that all the edges of the escaping chain of the horizontal side of T are parallel. Thus, in this case the dual tropical curve splits off a component. Assume now that the first edge of γ is even. According to Lemma 12, this edge is vertical. In addition, it is of length 2. If the second edge of γ is odd, then the same arguments as in the case $m = 2$ show the reducibility of the tropical curve dual to τ. Suppose that the second edge of γ is even. If this edge is non-vertical, it is horizontal by Lemma 12, and a component again is split off the tropical curve; see Figure 5(a). Thus, it remains to consider the case when the first three edges of γ are of integer length 2, the first and the second edges are vertical, and the third edge is non-vertical. According to Lemma 12, the third edge of γ is either horizontal or of slope 1; see Figures 5(c) and 5(d). In the former case the tropical curve splits off a component, and in the latter case the third edge cannot generate an escaping chain in the bottom direction.
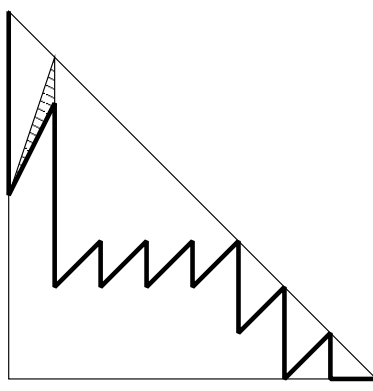
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Figure 6: Path γ for $\Sigma = \mathbb{P}^2$ and $m = 2$

To prove the asymptotic relations of Theorem 7, we slightly modify the constructions used in the proof of Theorem 3. For instance, if $m = 3$ and $\Sigma = \mathbb{P}^2$, we start the lattice path with two vertical edges $[(0, d), (0, d-2)]$ and $[(0, d-2), (0, d-4)]$ of length 2, take the third edge $[(0, d-4), (1, d-2)]$ of integer length 1 supplied with the triangle with vertices $(0, d-4)$, $(1, d-2)$ and $(1, d-1)$ (see Figure 6), and then continue the path as described in the proof of Theorem 3.

The final statement of Theorem 4 follows from Welschinger's wall crossing formula and the positivity of the Welschinger invariant of $P_1, P_2,$ and P_3 for the linear system corresponding to $\Pi_D \setminus \{x + y < 2\}$. $\square$

3.3.5 Auxiliary bound

The following lemma is proved exactly in the same way as Theorem 3.

Lemma 13 *The following inequality holds:*

$$\log W_{Q,D,0} \geq 2(d_1 + d_2) \log d_1 + O(d_1 + d_2), \quad \min\{d_1, d_2\} \to +\infty \ ,$$

where D is a divisor of bi-degree (d_1, d_2) on $Q = \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

3.3.6 Proof of Theorem 8

We prove at the same time the both parts: the positivity statement and the lower bound.

Pick an integer $n \geq 1$. In this proof we represent the linear system $|nD|$ on Q by a rectangle Π with vertices $(0, 0), (nd_1, 0), (nd_1, nd_2), (0, nd_2)$. According to Lemma 11, for any $m(n)$-element set $\mathcal{I} \subset \{1, \ldots, 2(nd_1 + nd_2) - 1 - m(n)\}$ and any $(\lambda^0, \mathcal{I})$-admissible path γ in Π, all the γ-consistent subdivisions of Π have positive

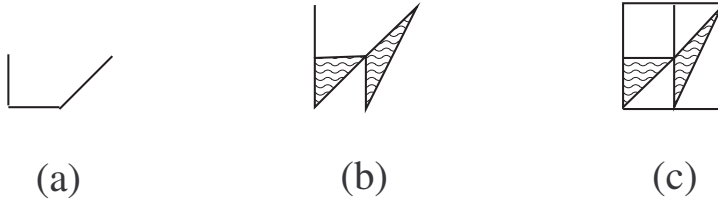(a)   (b)   (c)

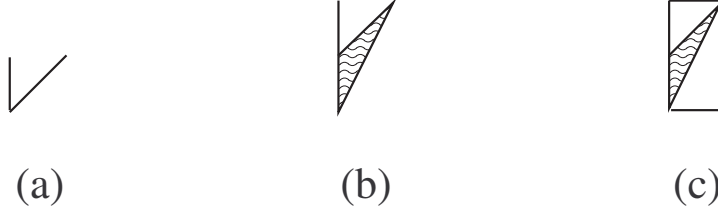Figure 7: Case of nd_1 and nd_2 even

(a)   (b)   (c)

Figure 8: Case of nd_1 odd and nd_2 even

Welschinger coefficients. Thus, to prove the theorem, it is sufficient to construct an appropriate number of consistent subdivisions.

Our construction ramifies in few cases. We use the fact that, as it follows from Lemma 12, if $1 \notin \mathcal{I}$ then the first edge of any $(\lambda^0, \mathcal{I})$-admissible path producing a consistent subdivision is vertical.

Suppose, first, that $m(n) = nd_1 + nd_2 - 1$, and put $\mathcal{I} = \{2, \ldots, m(n) + 1\}$.

If nd_1 and nd_2 are even, we consider the paths γ of the form $\gamma = \gamma_1 \cup \gamma_2$, where γ_1 consists of three edges of integer length 1 and looks like in Figure 7(a), and γ_2 is obtained by multiplication by 2 (and translation) of a $(\lambda^0, \emptyset)$-admissible path γ_2' related to divisors of bi-degree $(nd_1/2 - 1, nd_2/2)$ and providing a consistent subdivision. To obtain the set S_0 we complete the second and the third edges of γ_1 by primitive triangles as is shown in Figure 7(b). Then, we complete S_0 to γ-consistent subdivisions of Π which start from a pattern shown in Figure 7(c). Namely, the vertical strip of width 2 below the described square is filled up by rectangles 1×2 (*i.e.*, of width 1 and height 2), and the remaining rectangle $(nd_1 - 2) \times nd_2$ is filled up by the doubles of the elements of γ_2'-consistent subdivisions.

If nd_1 is odd and nd_2 is even, we consider the paths γ of the form $\gamma = \gamma_1 \cup \gamma_2$, where γ_1 consists of two edges of integer length 1 and looks like in Figure 8(a), and γ_2 is obtained by multiplication by 2 (and translation) of a $(\lambda^0, \emptyset)$-admissible path γ_2' related to divisors of bi-degree $((nd_1 - 1)/2, nd_2/2)$ and providing a consistent subdivision. To obtain the set S_0 we complete the second edge of γ_1 by a primitive triangle as is shown in Figure 7(b). Then, we complete S_0 to γ-consistent subdivisions of Π which start from a pattern shown in Figure 8(c). Namely, the vertical strip of width 1 below the described pattern is filled up by rectangles 1×2, and the
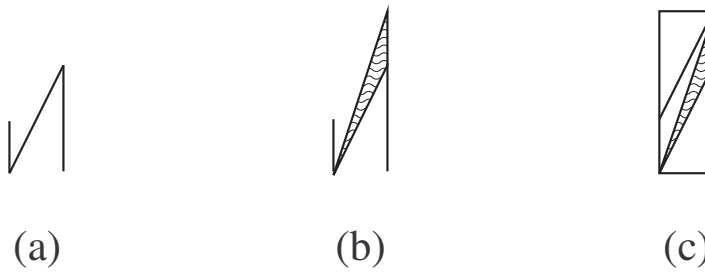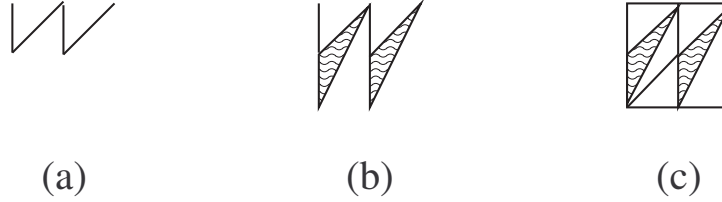
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Figure 9: Case of nd_1 and nd_2 odd

Figure 10: Case of odd m

remaining rectangle $(nd_1 - 1) \times nd_2$ is filled up by the doubles of the elements of γ_2'-consistent subdivisions.

If nd_1 and nd_2 are odd, we put $\mathcal{I} = \{1, \ldots, m(n)-2, m(n), m(n)+1\}$ and consider the paths γ of the form $\gamma = \gamma_1 \cup \gamma_2$, where γ_2 consists of three edges (two edges of integer length 1 and one edge of integer length 2) and looks like in Figure 9(a), and γ_1 is obtained by multiplication by 2 (and translation) of a $(\lambda^0, \emptyset)$-admissible path γ_1' related to divisors of bi-degree $((nd_1 - 1)/2, (nd_2 - 1)/2)$ and providing a consistent subdivision. To obtain the set S_0 we complete the second edge of γ_2 by a primitive triangle as is shown in Figure 9(b). Then, we complete S_0 to γ-consistent subdivisions of Π in the following way: we put the doubles of the elements of γ_1'-consistent subdivisions in the rectangle with vertices $(0, 1)$, $(nd_1, 1)$, (nd_1, nd_2), and $(0, nd_2)$; continue with the pattern shown in Figure 9(c); and fill up the strip situated to the left (resp., above) of the pattern by rectangles 2×1 (resp., 1×2).

Suppose now that $m(n) < nd_1 + nd_2 - 1$, and put $\mathcal{I} = \{j, j+2, j+3, \ldots, j+m(n)\}$, where $j = 2(nd_1 + nd_2 - m(n)) - 1$.

If $m(n)$ is odd, we consider the paths γ of the form $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

- γ_1 is the translation of a $(\lambda^0, \emptyset)$-admissible path γ_1' providing a consistent subdivision and related to divisors of nonnegative bi-degree $(nd_1 - 2b_1, nd_2 - 2b_2)$ with b_1 and b_2 nonnegative integers such that $2b_1 + 2b_2 = m(n) + 1$,

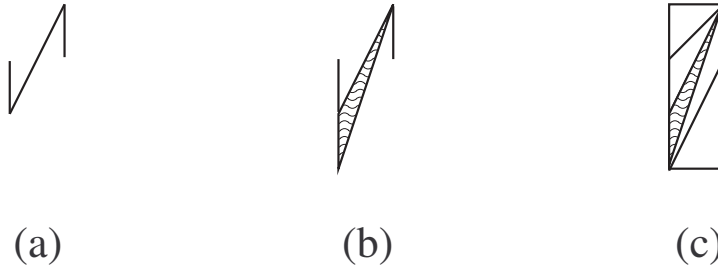- γ_2 consists of four edges of integer length 1 and looks like in Figure 10(a),

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Figure 11: Case of even $m(n)$

- γ_3 is obtained by multiplication by 2 (and translation) of a $(\lambda^0, \emptyset)$-admissible path γ_3' providing a consistent subdivision and related to divisors of bi-degree $(b_1 - 1, b_2)$.

To obtain the set S_0 we complete the second and the forth edges of γ_2 by primitive triangles as in Figure 10(b). Then, we complete S_0 to γ-consistent subdivisions of Π in the following way: we shift the elements of γ_1'-consistent subdivisions in the rectangle with vertices $(0, 2b_2)$, $(nd_1 - 2b_1, 2b_2)$, $(nd_1 - 2b_1, nd_2)$, and $(0, nd_2)$; superpose over γ_2 the pattern shown on Figure 9(c); put the doubles of the elements of γ_3'-consistent subdivisions in the rectangle with vertices $(nd_1 - 2b_1 + 2, 0)$, $(nd_1, 0)$, $(nd_1, 2b_2)$, and $(nd_1 - 2b_1 + 2, 2b_2)$; and fill up the remaining parts by rectangles of sizes 1×1, 1×2, and 2×1.

If $m(n)$ is even, we consider the paths γ of the form $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

- γ_1 is the translation of a $(\lambda^0, \emptyset)$-admissible path γ_1' providing a consistent subdivision and related to divisors of nonnegative bi-degree $(nd_1 - 2b_1 - 1, nd_2 - 2b_2)$ with b_1 and b_2 nonnegative integers such that $2b_1 + 2b_2 = m(n)$,

- γ_2 consists of three edges of integer length 1 and looks like in Figure 11(a),

- γ_3 is obtained by multiplication by 2 (and translation) of a $(\lambda^0, \emptyset)$-admissible path γ_3' providing a consistent subdivision and related to divisors of bi-degree (b_1, b_2).

To obtain the set S_0 we complete the second edge of γ_2 by a primitive triangle as in Figure 11(b). The further construction of γ-consistent subdivisions of Π uses the pattern shown on Figure 11(c) and is performed in the same way as in the case of odd $m(n)$.

The existence of γ-consistent subdivisions implies the positivity of the Welschinger invariants. To prove the required asymptotic inequality we bound from below, using Lemma 13, the number of constructed subdivisions. For example, in the very last case we take b_i, $i = 1, 2$, close to $nd_i \cdot \frac{m(n)}{2n(d_1 + d_2)}$ and verifying the conditions imposed in the construction, and then deduce from Lemma 13 that

$$\log W_{(nd_1 - 2b_1 - 1, nd_2 - 2b_2), 0} \geq (2nd_1 + 2nd_2 - 2\mu n) \log n + O(n + \psi(n) \log n),$$

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if $\mu < d_1 + d_2$, and

$$\log W_{(b_1,b_2),0} \geq \mu n \log n + O(n + \psi(n) \log n),$$

if $\mu > 0$. Using the positivity of the Welschinger coefficients we obtain

$$\log W_{D,m(n)} \geq \log W_{(nd_1-2b_1-1,nd_2-2b_2),0} + \log W_{(b_1,b_2),0}$$
$$\geq (2d_1 + 2d_2 - \mu)n \log n + O(n + \psi(n) \log n).$$

$\square$

References

[1] Di Francesco, P., Itzykson, C.: Quantum intersection rings. *in "The moduli space of curves"* (Texel Island, 1994), 81–148, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[2] Göttsche, L, and Pandharipande, A.: The quantum cohomology of blow-ups of $\mathbb{P}^2$ and enumerative geometry. *J. Diff. Geom.* **48** (1998), no. 1, 61–90.

[3] Itenberg, I., Kharlamov, V., and Shustin, E.: Welschinger invariant and enumeration of real rational curves. *International Math. Research Notices* **49** (2003), 2639–2653.

[4] Kontsevich, M. and Manin, Yu.: Gromov-Witten classes, quantum cohomology and enumerative geometry. *Commun. Math. Phys.* **164** (1994), 525-562.

[5] Mikhalkin, G.: Counting curves via the lattice paths in polygons. *C. R. Acad. Sci. Paris, Sér. I*, **336** (2003), no. 8, 629–634.

[6] Mikhalkin, G.: *Enumerative tropical algebraic geometry in $\mathbb{R}^2$*. Preprint arXiv: math.AG/0312530.

[7] Shustin, E.: *Patchworking singular algebraic curves, non-Archimedean amoebas and enumerative geometry*. Preprint arXiv: math.AG/0211278.

[8] Shustin, E.: *A tropical calculation of the Welschinger invariants of real toric Del Pezzo surfaces*. Preprint arXiv:math.AG/0406099.

[9] Viro, O.: *Dequantization of Real Algebraic Geometry on a Logarithmic Paper*. Proceedings of the 3rd European Congress of Mathematicians, Birkhäuser, Progress in Math, 201, (2001), 135–146.

[10] Welschinger, J.-Y.: Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry. *C. R. Acad. Sci. Paris, Sér. I,* **336** (2003), 341–344.

[11] Welschinger, J.-Y.: *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Preprint arXiv: math.AG/0303145.

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